

Basics of Hamiltonian Mechanics,

Liouville's Theorem

Poincaré Recurrence

→ Basics of Hamiltonian Mechanics

- Why?

L.E.: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$ → 2nd order eqn. for gen. coords.

$\left\{ \begin{array}{l} \frac{d}{dt} (p_i) = \frac{\partial L}{\partial q_i} \\ \downarrow \\ \text{generalized momentum} \end{array} \right\}$

H.E.: $\dot{q} = -\partial H / \partial p$ → 2 first order equations

$\dot{p} = \partial H / \partial q$ → coordinates and momenta are equal footing in fact interchangeable...

H.E. very useful for phase space descriptions, formulations.

N.B.: History:

- Lagrange, 1756 (France)
 - ⇒ minimization ↔ Fermat
- Hamilton, 1823 (Ireland)
 - ⇒ outgrowth of ray tracing using Huygens' principle (rays orthogonal)

- Formulation: Legendre transformation $\dot{q} \rightarrow p$

In general $L = L(q, \dot{q})$

n.b. t is parameter

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}$$

$$= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) dq + \frac{\partial L}{\partial \dot{q}} d\dot{q}$$

$$= \dot{p} dq + p d\dot{q}$$

$$d(p\dot{q}) = p d\dot{q} + \dot{q} dp$$

$$dL = \dot{p} dq + d(p\dot{q}) - \dot{q} dp$$

$$d(p\dot{q} - L) = -\dot{p} dq + \dot{q} dp$$

$$= dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$

n.b. $H = H(q, p)$

- Legendre transform via construction

~~$$\frac{\partial H}{\partial \vec{x}} = \begin{pmatrix} \dot{z} \\ \dot{p} \end{pmatrix}$$~~

so, equating

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{z} = \frac{\partial H}{\partial p}$$

Hamiltonian
EOMs
q, p etc.

→ Hamiltonian is function of generalized coordinates and momenta. (Very const.)

→ to construct Hamiltonian formulation, need not have conservative system. Need only be able to invert:

$$p_i = \frac{\partial L}{\partial \dot{z}_i} \quad \text{to solve } \dot{z}_i \text{ in terms } p_i$$

N.B.: Conservation?

- in Lagrangian mechanics,

$$E = \dot{z} \frac{\partial L}{\partial \dot{z}} - L \quad \Rightarrow \text{linked time translation symmetry}$$

$$\partial_t L = 0 \Rightarrow \dot{E} = 0$$

Now, in Hamiltonian Mechanics:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial z} \dot{z} + \frac{\partial H}{\partial p} \dot{p}$$

$$= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial z} \left(\frac{-\partial H}{\partial p} \right) + \frac{\partial H}{\partial p} \left(\frac{\partial H}{\partial z} \right)$$

$$= \partial H / \partial t.$$

Thus, if no explicit time dependence, energy conserved ($E = \sum_j \frac{\partial L}{\partial \dot{z}_j} - L$, $H = p \dot{z} - L$)

well $H = \text{const.}$ (equiv. to $\partial L / \partial t = 0$ in Lagrangian formulation)

→ Constructing Hamiltonians.

→ trivial.

Particle moves in $U(r, \theta, \phi)$. Construct Hamiltonian \mathcal{H}

$$L = T - U$$

$$= \frac{1}{2} m \left(\frac{ds}{dt} \right)^2 - U$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$\underline{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U$$

Now, $H = \underline{p} \cdot \underline{\dot{q}} - L$

$$= p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

and

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$$

so $\dot{r} = p_r / m$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

so $\dot{\theta} = p_\theta / m r^2$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi}$$

so $\dot{\phi} = p_\phi / m r^2 \sin^2 \theta$

Now need eliminate generalized velocities

$$H = p_r \left(\frac{p_r}{m} \right) + p_\theta \left(\frac{p_\theta}{mr^2} \right) + p_\phi \left(\frac{p_\phi}{mr^2 \sin^2 \theta} \right)$$

$$= \left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} \right) + U$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + U$$

and Hamiltonian EOMs follow.

→ off-beat

$$L(\theta, \dot{\theta}) = e^{\dot{\theta}}$$

$$H = p \dot{\theta} - L$$

$$\text{Now } p = \frac{\partial L}{\partial \dot{\theta}} = e^{\dot{\theta}} \Rightarrow \dot{\theta} = \ln p$$

$$H = p \ln p - p$$

$$\dot{\theta} = \ln p, \quad \dot{p} = 0$$

→ time dependent

$$L = \frac{1}{2} G(q, t) \dot{q}^2 + F(q, t) \dot{q} - V(q, t)$$

Again: → $H = p\dot{q} - L$

$$\rightarrow p = \frac{\partial L}{\partial \dot{q}}$$

$$p = G(q, t) \dot{q} + F(q, t)$$

$$\dot{q} = (p - F(q, t)) / G(q, t)$$

⇒

$$\begin{aligned} H &= p \left(\frac{p-F}{G} \right) - \frac{G}{2} \left(\frac{p-F}{G} \right)^2 - F \left(\frac{p-F}{G} \right) + V \\ &= \frac{(p-F)^2}{2G} + V \end{aligned}$$

and H EoMs follow.

N.B. $\left. \begin{aligned} F &= F(q, t) \\ G &= G(q, t) \end{aligned} \right\} \text{ here}$

→ in general, Hamiltonian Formulation requires invertibility of generalized velocities in terms of generalized momenta.

c.e. need solve $p_i = \frac{\partial L}{\partial \dot{z}_i}$ for $\dot{z}_i(p, z)$

to eliminate \dot{z}_i

Generally,

$$p_i = \frac{\partial L}{\partial \dot{z}_i}$$

locally,

$$dp_i = d\left(\frac{\partial L}{\partial \dot{z}_i}\right) = \frac{\partial^2 L}{\partial \dot{z}_i \partial \dot{z}_j} d\dot{z}_j$$

⇒

$$dp_i = A_{ij} d\dot{z}_j, \quad A_{ij} = \frac{\partial^2 L}{\partial \dot{z}_i \partial \dot{z}_j}$$

$$d\dot{z}_j = A_{ij}^{-1} dp_i \quad \text{and can solve and eliminate}$$

obviously, - A_{ij} must be invertible
- $\det A_{ij} \neq 0$ required!

IF $\det A_{ij} = 0 \Rightarrow$ special constraint exists, requiring treatment by Dirac brackets, instead Poisson brackets. Via that approach, can still formulate Hamiltonian.

Ex. \rightarrow Give an example of a system for which a conventional Hamiltonian cannot be formulated. Explain why.

Non-trivial example: (cf Dirac lectures, '64)

Consider a charged particle in x - y plane, in magnetic field $B_0 \hat{z}$.

\rightarrow Strong field limit

$$\Rightarrow L = \frac{1}{2} m v^2 + \frac{q}{c} \underline{v} \cdot \underline{A} - U$$

$$\underline{A} = \frac{B_0}{2} \hat{z} \times \underline{r} = \frac{B_0}{2} (x \hat{y} - y \hat{x})$$

so can re-scale as:

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{q B_0}{2c} (x \dot{y} - y \dot{x}) - \frac{q B_0}{2c} U$$

and re-scale by m to obtain:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{eB_0}{2mc}(x\dot{y} - y\dot{x}) - \frac{eB_0}{2mc}u(x,y)$$

Now $\eta \equiv \frac{eB_0}{2mc} = \frac{B_0}{2} \text{ cycl.}$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \eta(x\dot{y} - y\dot{x}) - \eta u(x,y)$$

Now, consider $\eta \gg \frac{d}{dt}$, so

$$\eta(x\dot{y} - y\dot{x}) \gg \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

N.B. - strong field limit, i.e. drop kinetic energy.
 - here Lagrangian linear in velocity \Rightarrow obvious difficulty in inversion!

i.e. $L \approx \eta(x\dot{y} - y\dot{x}) - \eta u(x,y)$

L EOMS: $\frac{d}{dt}(-y) = -\partial u / \partial x$
 $\frac{d}{dt}(x) = -\partial u / \partial y$

Now, For Hamiltonian:

$$P_x = -my = \partial L / \partial \dot{x}$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x}$$

and no inversion of \dot{x}, \dot{y} in terms
 P_x, P_y possible! \rightarrow special feature
 is strong
 field constraint

Further:

$$H = P_x \dot{x} + P_y \dot{y} - L$$

$$= \dot{x}(-my) + \dot{y}(m\dot{x}) - m(x\dot{y} - y\dot{x}) + mU$$

$$= mU \quad (\text{akin G.C. Plasma})$$

Momenta drop out!

What is the problem here?

- Lagrangian linear in V

- Coordinates (q 's) and momenta
 (p 's) not independent.

→ Need attack by adding constraint ~~to usual story~~ to usual story.

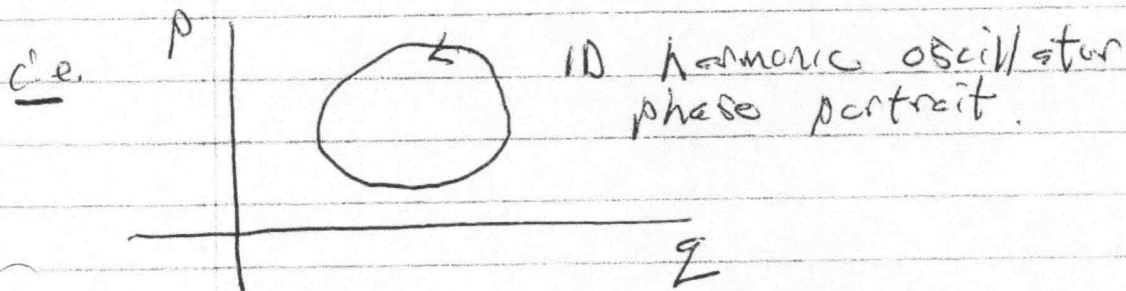
TBC.

→ Using Hamiltonians

- by treating q, p symmetrically, Hamiltonians are natural variables for phase space description of dynamics

i.e. → replace \blacksquare 2nd order Lagrange equation with 2 first order Hamilton eqn.

→ natural for describing phase space flow



in order describing phase space dynamics, need:

- phase space density $\rho(\underline{q}, \underline{p})$ and its evolution

c.e. $F(\underline{q}, \underline{p}) \Rightarrow \rho(\underline{q}, \underline{p})$

↓
distribution function

$$\langle E_k \rangle = \int d^{3N} \underline{p} \int d^{3N} \underline{q} \frac{p^2}{2m} F(\underline{q}, \underline{p}) / \int_{\Gamma}$$

- understanding of nature of the flow.

Now, if $\underline{V}_\Gamma = (\dot{\underline{q}}, \dot{\underline{p}})$

↓
phase space flow (2ND dim. vector)

then $\int_{\Gamma} \underline{V}_\Gamma = 0 \Rightarrow$ flow is incompressible

c.e.

$$\frac{\partial}{\partial \underline{q}} \dot{\underline{q}} + \frac{\partial}{\partial \underline{p}} \dot{\underline{p}} = \frac{\partial}{\partial \underline{q}} \frac{\partial H}{\partial \underline{p}} + \frac{\partial}{\partial \underline{p}} \left(\frac{-\partial H}{\partial \underline{q}} \right)$$

= 0

consequence only of Hamiltonian structure!

⇒ generic to Hamiltonian structure, phase space flow is incompressible,

⇒ phase space density conserved along particle trajectories

c.e. for particles not created or destroyed,

$$\frac{\partial \rho}{\partial t} + \nabla_{\Gamma} \cdot (\rho \underline{V}) = 0 \quad \left\{ \begin{array}{l} \text{phase space} \\ \text{continuity} \end{array} \right.$$

c.e.

$$\frac{\partial \rho}{\partial t} + \sum_i \left\{ \frac{\partial}{\partial z_i} \cdot (\dot{z}_i \rho) + \frac{\partial}{\partial p_i} \cdot (\dot{p}_i \rho) \right\} = 0$$

so

$$\frac{\partial \rho}{\partial t} + \underline{V} \cdot \nabla_{\Gamma} \rho + \rho \nabla_{\Gamma} \cdot \underline{V} = 0$$

and

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\dot{z}_i \frac{\partial}{\partial z_i} \rho + \dot{p}_i \frac{\partial}{\partial p_i} \rho \right) + \sum_i \rho \left(\frac{\partial}{\partial z_i} \cdot \dot{z}_i + \frac{\partial}{\partial p_i} \cdot \dot{p}_i \right) = 0$$

phase
space

5.

For Hamiltonian system:

$$\nabla \cdot \underline{V}_H = 0 \Leftrightarrow \text{phase space flow incompressible}$$

Phase volume conserved!

Liouville's Thm.

50

$$\frac{\partial \rho}{\partial t} + \underline{V}_H \cdot \nabla_H \rho = 0$$

⇒ Phase space density conserved along particle trajectories.

⇒ Locally conserved phase space density.

n.b. For N particle system:

$$\rho = \rho(\underline{p}_1, \underline{q}_1, \dots, \underline{p}_N, \underline{q}_N) \rightarrow N \text{ body distribution fn.}$$

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^N \left(\underline{q}_i \cdot \frac{\partial}{\partial \underline{q}_i} + \underline{p}_i \cdot \frac{\partial}{\partial \underline{p}_i} \right) \rho = 0$$

if dilute, etc. can derive:
Boltzmann Eqn. (via BBGK y hierarchy)

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \underline{\nabla} F + \underline{q} \cdot \frac{\partial F}{\partial \underline{v}} = C(F, F)$$

collision operator
(\rightarrow 2 body interaction)

if collisionless:

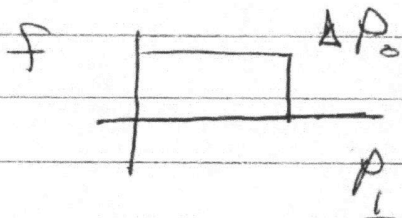
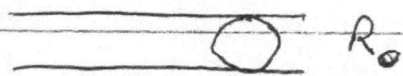
$$\frac{\partial F}{\partial t} + \underline{v} \cdot \underline{\nabla} F + \underline{q} \cdot \frac{\partial F}{\partial \underline{v}} = 0$$

Vlasov equation

Example:

Consider a particle beam, with transverse momentum dispersion $\Delta_{\perp} p_0$, and radius R_0 . Comment on what will happen if attempt to focus to $R, \ll R_0$.

Consider beam as Hamiltonian system,

de

Key:

- phase space volume conserved
- conservative irrelevant!
- no a priori connection of conservative dynamics and Hamiltonian structure

$$V_{\text{in}} \Big|_{\text{before focus}} = V_{\text{out}} \Big|_{\text{after focus}}$$

$$\pi R_0^2 \pi (\Delta P_{\perp 0})^2 = \pi R_1^2 \pi (\Delta P_{\perp 1})^2$$

$$\Rightarrow \Delta P_{\perp 1} = \frac{R_0}{R_1} \Delta P_{\perp 0}$$

so dispersion increases, to compensate reduction in spatial focal point region.

\Rightarrow inefficient.

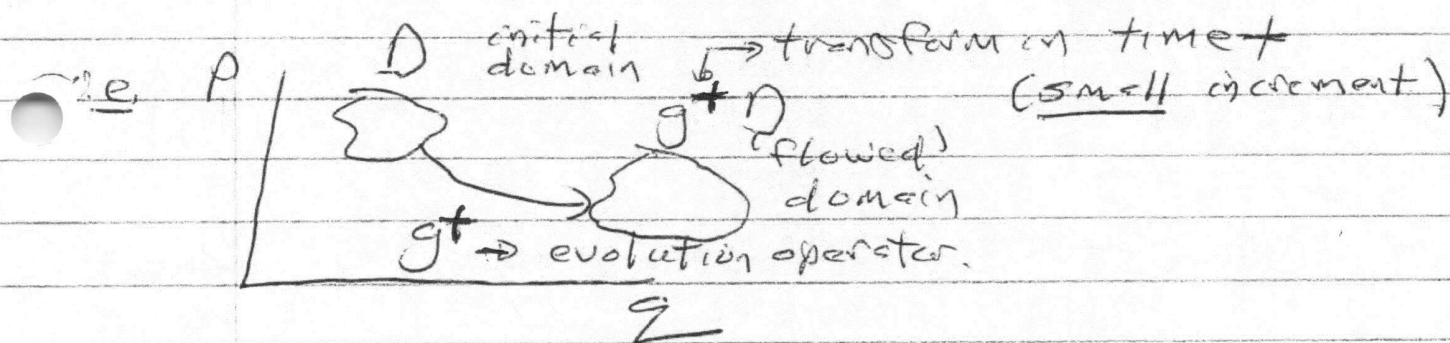
Poincaré Recurrence Theorem

another take on phase space flow,
Liouville's theorem:

Define phase flow g^t : transformation

s/t

$\underline{p}(0), \underline{q}(0) \rightarrow \underline{p}(t), \underline{q}(t)$ along
Hamiltonian trajectories.



Now:

Hamiltonian eqns constitute
autonomous system

i.e. $\dot{\underline{x}} = \underline{F}(\underline{x})$

$$\underline{v}_H = \begin{pmatrix} \partial H / \partial \underline{p} \\ -\partial H / \partial \underline{q} \end{pmatrix} = \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix}$$

then, for small increment:

$$\underline{g^t(x)} = \underline{x} + \underline{f(x)}t + o(t^2)$$

so then phase volume at t :

Jacobian of transform

$$V_{\phi}(t) = \int_{D(0)} dx \left| \frac{\partial x'}{\partial x} \right|$$

ϕ
metric
domain

$$= \int_{D(0)} dx \det \left| \frac{\partial g^t(x)}{\partial x} \right|$$

Now

$$\frac{\partial g^t(x)}{\partial x} = \underline{\underline{I}} + \frac{\partial f}{\partial x} t + o(t^2)$$

but now use identity (small t):

$$\det \left(\underline{\underline{I}} + \underline{\underline{A}}t \right) = 1 + t \operatorname{tr} \underline{\underline{A}} + \dots$$

so

$$V(t) = \int_{D(0)} d^3x \left[1 + \text{tr} \left[\frac{\partial \underline{F}}{\partial \underline{x}} \right] + O(t^2) \right]$$

but $\text{tr} \frac{\partial \underline{F}}{\partial \underline{x}} = \underline{D} \cdot \underline{F}$

from $\underline{v}_D = \underline{F}$, $\underline{D} \cdot \underline{F} = \underline{D}_D \cdot \underline{v}_D = 0$

\downarrow
 \underline{F} is phase space flow velocity

so, for t^2 ,

$V(t) = V(0)$
 as expected.

\Rightarrow phase volume conserved.

\Rightarrow no attractors in Hamiltonian mechanics i.e. no asymptotically stable positions, cycles.

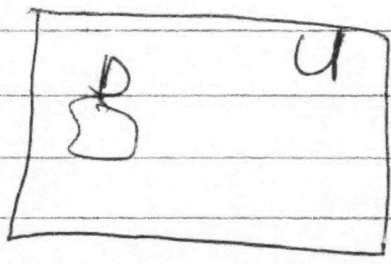
so come to:

Poincare Recurrence Theorem

- fundamental to ergodic theory
- inspiration for F. Nietzsche...

\Rightarrow Poincare's "what goes around, comes

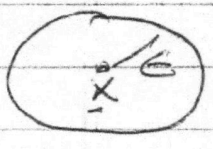
around, arbitrarily closely", for
bounded Hamiltonian system...
states



$U \equiv$ system universe,
bounded

g^t Hamiltonian, so
volume preserving

For any \underline{x} in U , can define
 $B(\underline{x}, \epsilon)$



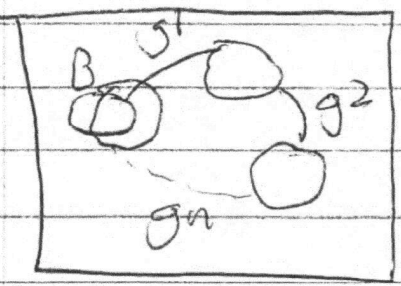
ball in phase space
around pt \underline{x} (ϵ, ϵ)
of radius ϵ

then $\exists \underline{x}' \in B(\underline{x})$ s.t.
 $g^n(\underline{x}') \in B(\underline{x})$

i.e. { there is a point in the ϵ -ball
of \underline{x} such that n iterations of
evolution operator yield a
point also in the ϵ ball.

i.e. { point recurs, arbitrarily
closely "

c.e.



consider $(g^n(B))$
 if each g^i disjoint

$\lim_{n \rightarrow \infty} \cup g^n \rightarrow \infty$, but U bounded
 \Rightarrow contradiction

so

$$g^k(B) \cap g^l(B) \neq \emptyset$$

intersection
 of
 arbitrary
 iterates
not empty.

$$\Rightarrow g^{k-l}(B) \cap B \neq \emptyset$$

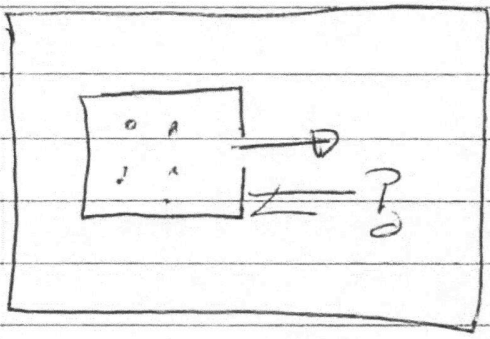
so

$$\exists \text{ some } x' \in g^{k-l}(B) \cap B$$

so there is some x' arbitrarily
 close to x

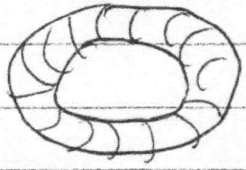
Q.E.D.

Implications:



box with particles,
 → particles escape thru hole
 → eventually, will go back in but may be a while

if
 torus



$$\psi_1 = \alpha_1$$

$$\psi_2 = \alpha_2$$

α_1 / α_2
irrational

then if $g^t (\psi_1, \psi_2) \rightarrow (\psi_1 + \alpha_1 t, \psi_2 + \alpha_2 t)$

α_1 / α_2 irrational \Rightarrow winding fills torus.

comes arbitrarily close

→ Poincare Recurrence - FAQ's :

- refs:

- V.I. Arnold, "Mathematical Methods of Classical Mechanics"

- S. Chandrasekhar "Stochastic Problems in Physics and Astronomy"
 Rev. Mod. Phys. 15, 1 (1942), online

- G. Zaslavsky "Hamiltonian Chaos and Fractional Dynamics"

- Why Care? (apart from interest)

- ergodic theory

$$\text{i.e. } \langle A \rangle_{\text{ensemble}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt A(t)$$

\downarrow
 ensemble avg. \iff time average

points :

- $B(x, \epsilon)$ \rightarrow trajectory returns arbitrarily closely to it.
range of ϵ

- any ensemble avg \implies partition \implies

⇒ coarse graining $\Delta p, \Delta x$

- time average guaranteed to fill the space, as will find $\Pi, -\Pi_2 <$

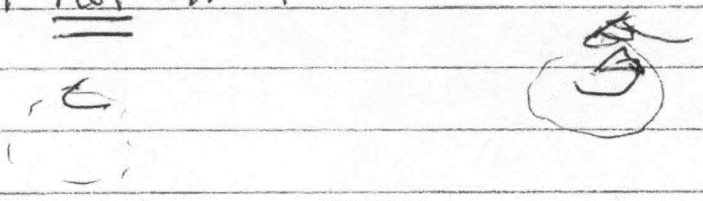
$\sqrt{(\Delta p)^2 + (\Delta x)^2}$

- what of harmonic oscillator?

nb - oscillator \neq limit cycle

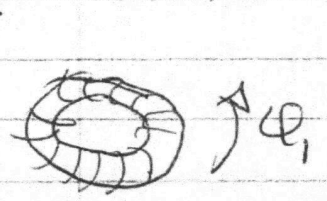
↓
closed trajectory
but not attractor

↓
attractor



* - closed, periodic trajectories are generally the exception (though surely possible)

i.e. consider toroidal surface



$\mathbb{S}^1 \times \mathbb{S}^1$

$\phi_1 = x_1$

$\phi_2 = x_2$

$\alpha_1/\alpha_2 \rightarrow$ rational \rightarrow closed cycle
 \rightarrow curve

$\alpha_1/\alpha_2 \rightarrow$ irrational $\rightarrow (g^t)^n$ winding fills surface, ^{iteration} some comes arbitrarily close to initial point.
 \Rightarrow surface

\downarrow $(g^t)^n$ n.b. # iterations \gg # actions.

time for recurrence is long.